Sought or Sold? Reflecting Motive in Insurance Pricing

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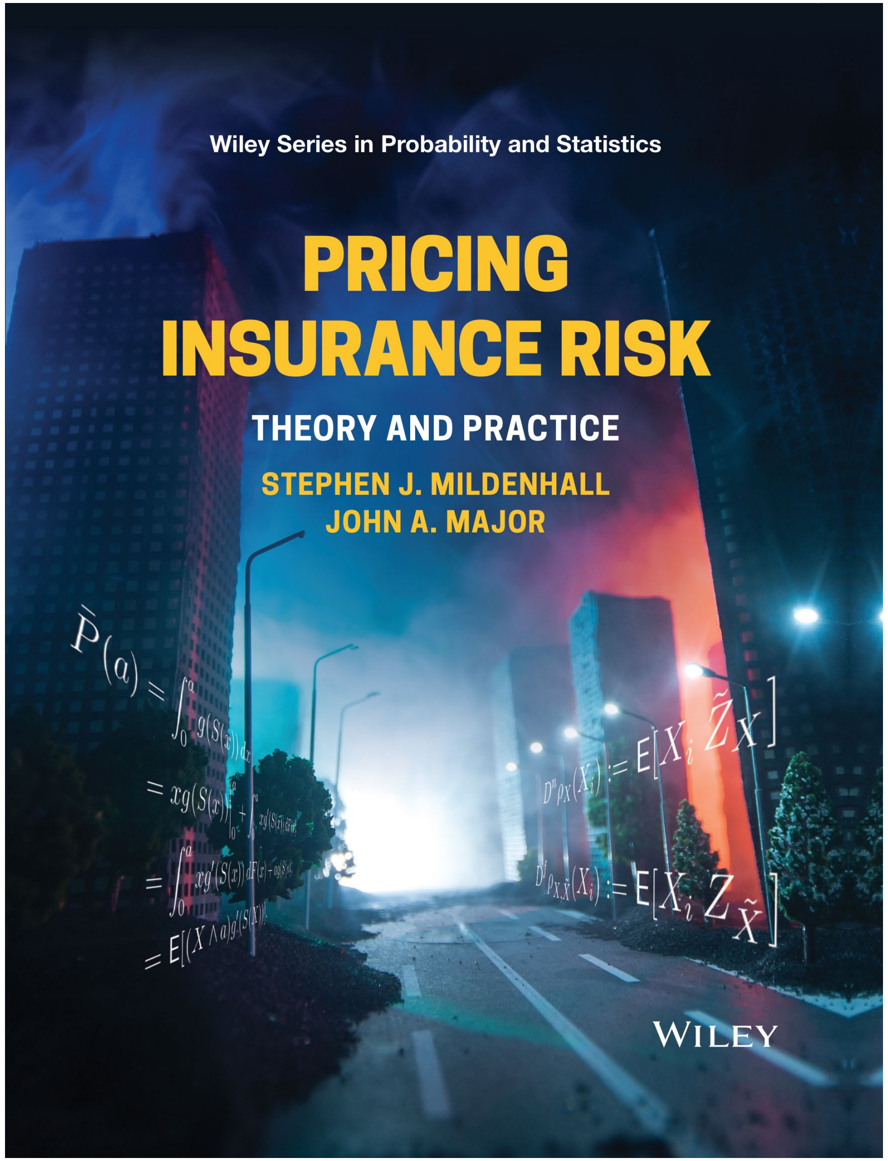
Abstract

The influence of buyer motivation on insurance pricing is under-explored. This talk delves into the dynamics of insurance policy cash flows and capital financing, demonstrating their primary divergence lies in buyer intent. Using Spectral Risk Measures, we'll unravel the pricing implications of such motivations within a one-period framework. These insights are built upon the foundational concepts introduced in "Pricing Insurance Risk" (Wiley 2022, co-authored with John Major) and are brought to life with examples using the speaker's open-source `aggregate` software package. To maximize engagement and understanding, attendees will be asked to opine on a pricing problem during the talk.



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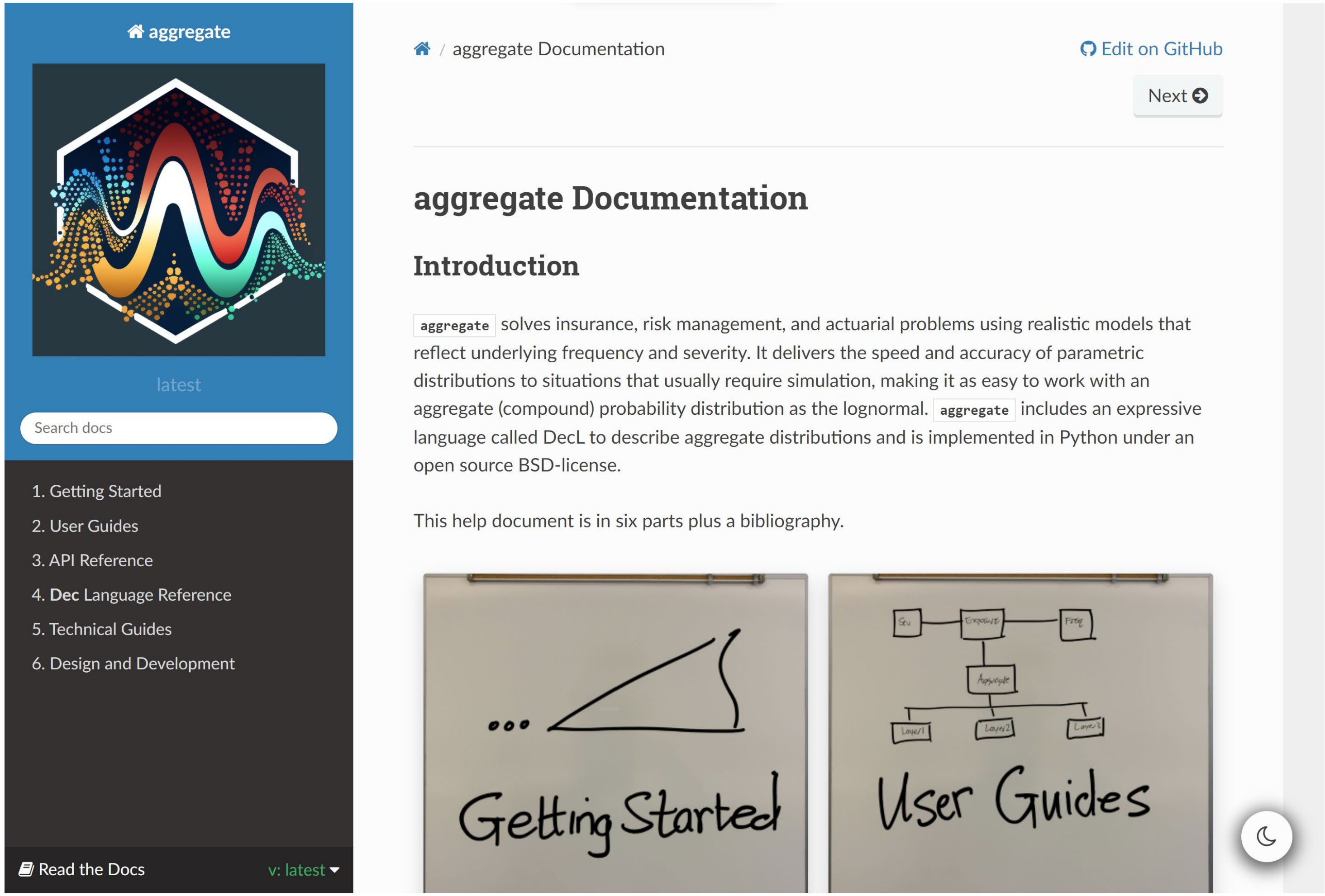


https://aggregate.readthedocs.io/en/latest/

Three Themes

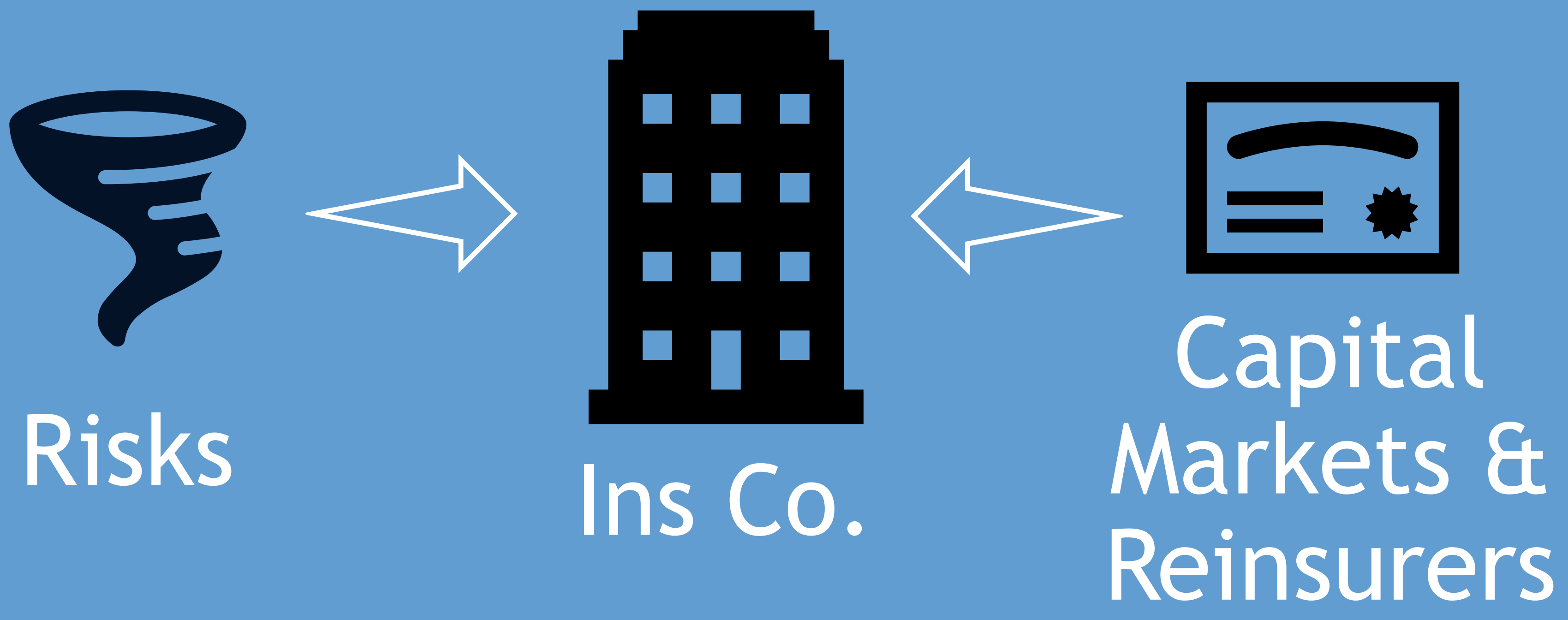
Equality: treat all an insurer's contingent cash flow contracts in the same way.

Motivation matters: is cash flow **sought** by or **sold** to the customer?

Spectral methods reflect motivation & connect value to risk appetite.

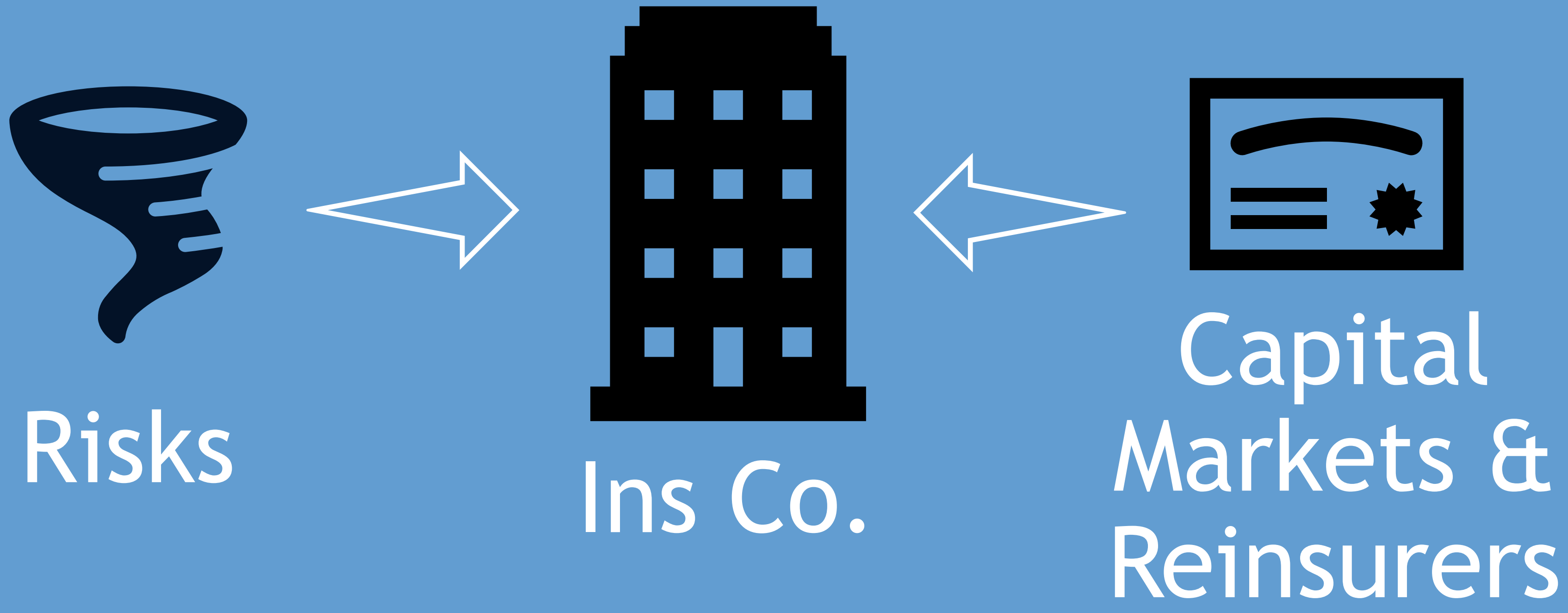
Market Setup

One-period insurer



$t = 0$	Premium →	← Collateral or Capital
$t = 1$	Loss payments ←	→ Residual collateral or assets

One-period insurer, no default*



$t = 0$	Premium →	← Collateral or Capital
$t = 1$	Loss payments ←	→ Residual collateral or assets

* Default is an important but irrelevant complication

Anchoring Exercise

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Ins Co. t = 1 Cash Flows

	X1	X2	X3	X4
0	36	0	29	35
1	40	0	25	35
2	28	0	37	35
3	22	0	43	35
4	33	7	25	35
5	32	8	25	35
6	31	9	25	35
7	45	10	10	35
8	25	40	0	35
9	25	75	0	0

- Cash flows from insurer to four different counter-parties at t = 1, all business written by Ins Co.
- Ten equally likely scenarios, 0-9, represent all possible outcomes
- Ignore investment income, taxes, expenses etc.

What is your target premium at t = 0 to pay each cash flow?

Extra credit: What does each cash flow represent?

https://forms.office.com/r/SBdWi7Qz3v

Discussion

Ins Co. t = 1 Cash Flows

	X1	X2	X3	X4	total
0	36	0	29	35	100
1	40	0	25	35	100
2	28	0	37	35	100
3	22	0	43	35	100
4	33	7	25	35	100
5	32	8	25	35	100
6	31	9	25	35	100
7	45	10	10	35	100
8	25	40	0	35	100
9	25	75	0	0	100

- Payments sum to 100 in every scenario
- No net risk
- Ignore investment income
- Total premium should be ≥ 100
- No net risk margin a possible solution

Ins Co. t = 1 Cash Flows

	X1	X2	X3	X4	total
0	36	0	29	35	100
1	40	0	25	35	100
2	28	0	37	35	100
3	22	0	43	35	100
4	33	7	25	35	100
5	32	8	25	35	100
6	31	9	25	35	100
7	45	10	10	35	100
8	25	40	0	35	100
9	25	75	0	0	100
EX	31.700	14.900	21.900	31.500	100
CV	0.215	1.545	0.623	0.333	0
Skew	0.456	1.791	-0.369	-2.667	0

- X_1, X_2 appear insurance-like
  - Moderate to high CV
  - Positive skewness
- X_1 non-cat line loss payments
  - Attritional payments in all scenarios
  - Moderate CV
- X_2 cat line loss payments
  - 40% chance of no payment
  - Extreme CV and skewness
- Insurance **sought** by buyer

Ins Co. t = 1 Cash Flows

	X1	X2	X3	X4	total
0	36	0	29	35	100
1	40	0	25	35	100
2	28	0	37	35	100
3	22	0	43	35	100
4	33	7	25	35	100
5	32	8	25	35	100
6	31	9	25	35	100
7	45	10	10	35	100
8	25	40	0	35	100
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- X_3, X_4 look capital or reinsurance-like
  - Negative correlation with $X_1 + X_2$
  - Negative skewness
- X_4 return of collateral on a 35 xs 65 collateralized aggregate cover
  - X_4 = 35 when no ceded loss
  - X_4 = 0 when limit loss
  - Compare with cat bond cash flows
- X_3 equity residual value
  - 100 minus sum of other cash flows
- Financing **sold** to buyer

Ins Co. t = 1 Cash Flows

	X1	X2	X3	X4	total	Gross	Ceded	Net	Financing
0	36	0	29	35	100	36	0	36	64
1	40	0	25	35	100	40	0	40	60
2	28	0	37	35	100	28	0	28	72
3	22	0	43	35	100	22	0	22	78
4	33	7	25	35	100	40	0	40	60
5	32	8	25	35	100	40	0	40	60
6	31	9	25	35	100	40	0	40	60
7	45	10	10	35	100	55	0	55	45
8	25	40	0	35	100	65	0	65	35
9	25	75	0	0	100	100	35	65	0
EX	31.700	14.900	21.900	31.500	100	46.600	3.500	43.100	53.400
CV	0.215	1.545	0.623	0.333	0	0.515			0.322
Skew	0.456	1.791	-0.369	-2.667	0	1.590			-0.788

- X_1 non-cat insurance
- X_2 cat insurance
- X_3 equity residual
- X_4 35 xs 65 reinsurance

- Gross = $X_1 + X_2$
- Ceded = $35 - X_4$
- Net = Gross – Ceded
- Financing = $X_3 + X_4$
- Gross + Financing = 100

Summary of Cash Flow Characteristics

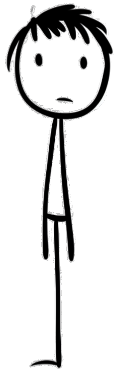
Characteristic	Insurance, risk assumption	Financing, risk bearing
Flow at t = 0	Fixed inflow	Fixed inflow
Flow at t = 1	Contingent outflow	Contingent outflow
Skewness	Positive	Negative
Margin, t=0 flow – E[t=1 flow]	Positive to Ins Co.	Negative to Ins Co.
Management	Underwriting / CUO Reinsurance	Finance / CFO
Motivation	**Initiated (sought) by insured**	**Initiated (sold) by insurer**

Motivation is the differentiating characteristic; it is invisible in cash flows

Target Premium or Price: Bid or Ask?

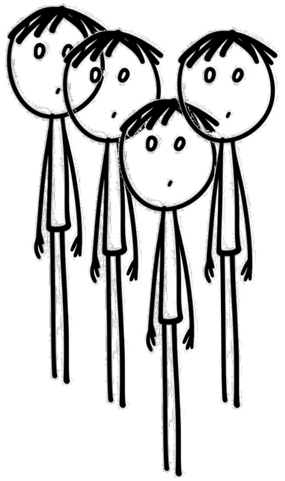
Bid Price and Ask Price: Transaction Uncertainty

Commodity Product



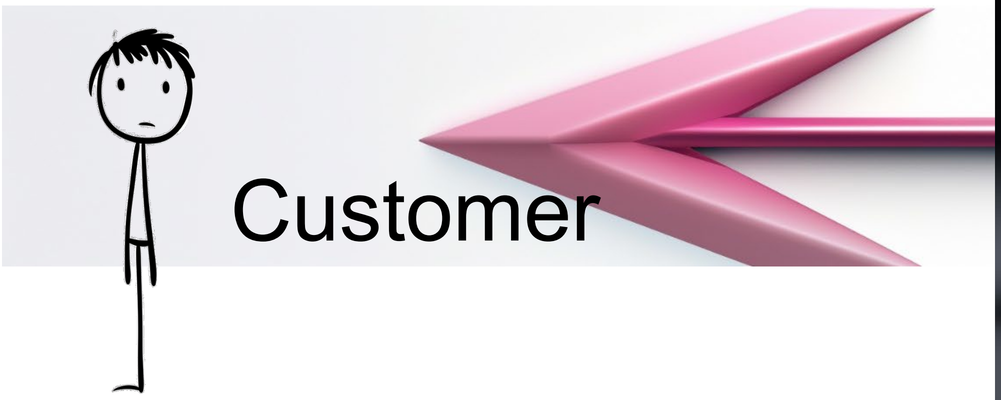
Customer

Market

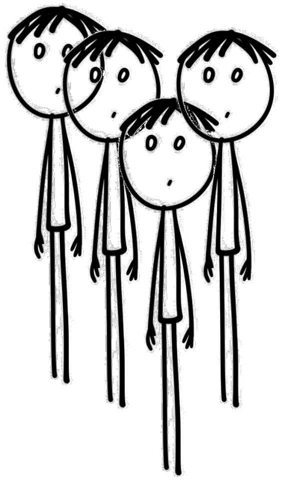


Uncertainty:
Buy or Sell?
Bid < Ask

Bid Price and Ask Price: Distinguish by Motivation



Specific Product Sold by Market



Is product **sought** by the customer (ask) or **sold** to the customer (bid)?

Weather Derivative Example

- Contract C pays $1 if temperature at [location] on [date] is **above** [strike]°C

- Pays with estimated objective probability (and hence loss cost) equal to *p*
- Quote price $\pi(C) > p$
- Think of $\pi(C)$ as a risk adjusted probability

- What is price for C' that pays $1 if temperature is **below** strike?

A weather derivative contract could pay $1 if the temperature in Central Park, NYC on July 1 2024 is above 35°

Weather Derivative Example

- What is price for C' that pays \$1 if temperature is **below** strike?

- Bundle of C + C' pays \$1 for sure and has cost (and value) equal to 1
- If π is a no arbitrage, additive pricing rule then $\pi(C + C') = \pi(C) + \pi(C') = 1$

- Implies $\pi(C') = 1 - \pi(C) < 1 - p$
- Suggests quoting C' under-cost

A weather derivative contract could pay \$1 if the temperature in Central Park, NYC on July 1 2024 is above 35°

Weather Derivative Example

- Why quote C' under cost? C and C' appear symmetric

Weather Derivative Example

- Why quote C' under cost? C and C' **appear** symmetric

- If C has been **sought** and if C' can be **sold** at any price ≥ 1 – π(C) then Ins Co. makes an arbitrage (riskless) profit

- Ins Co. could quote C' under cost, if it is being **sold** as part of financing

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Weather Derivative Example

- Contract C' pays $1 if temperature is **below** [strike]°C

- C' is **sought** by buyers desiring cover
- Ins Co. wants to quote a price with a positive margin

- Solution: $1 = \pi(C + C') \le \pi(C) + \pi(C')$
- π is sub-additive
- How can we ensure no arbitrage?

Some entities benefit from hot weather and some from cold, driving natural demand for both C and C'

Finance for Insurance

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Pricing rules with motivation

- Motivation requires two pricing rules
  - A(X): **ask** price for X when **sought** by the buyer
  - B(X): **bid** price for X when **sold** to the buyer
  - Same X in both cases

- **No arbitrage condition**

X sought, −X sold } net X − X = 0 is risk-free, zero value ⇒ $A(X) + B(-X) = 0$

$$A(X) = -B(-X)$$
$$B(X) = -A(-X)$$

If both X and −X are sought, there is a **risky** (non-arbitrage) profit

Efficient market pricing rules usually use a state-price density

- $P(X) = E[XZ]$
  - X is a random variable giving cash flow in each state of the world
  - $Z \geq 0$, $E[Z] = 1$ is **state price density**, giving the market price of \$1 in each state
- P is linear in X and has no bid/ask spread

$$-P(-X) = -E[-XZ] = E[XZ] = P(X)$$

- How can P be adjusted to include a spread?

A point in an economist's state-space reflects a possible future state of the world

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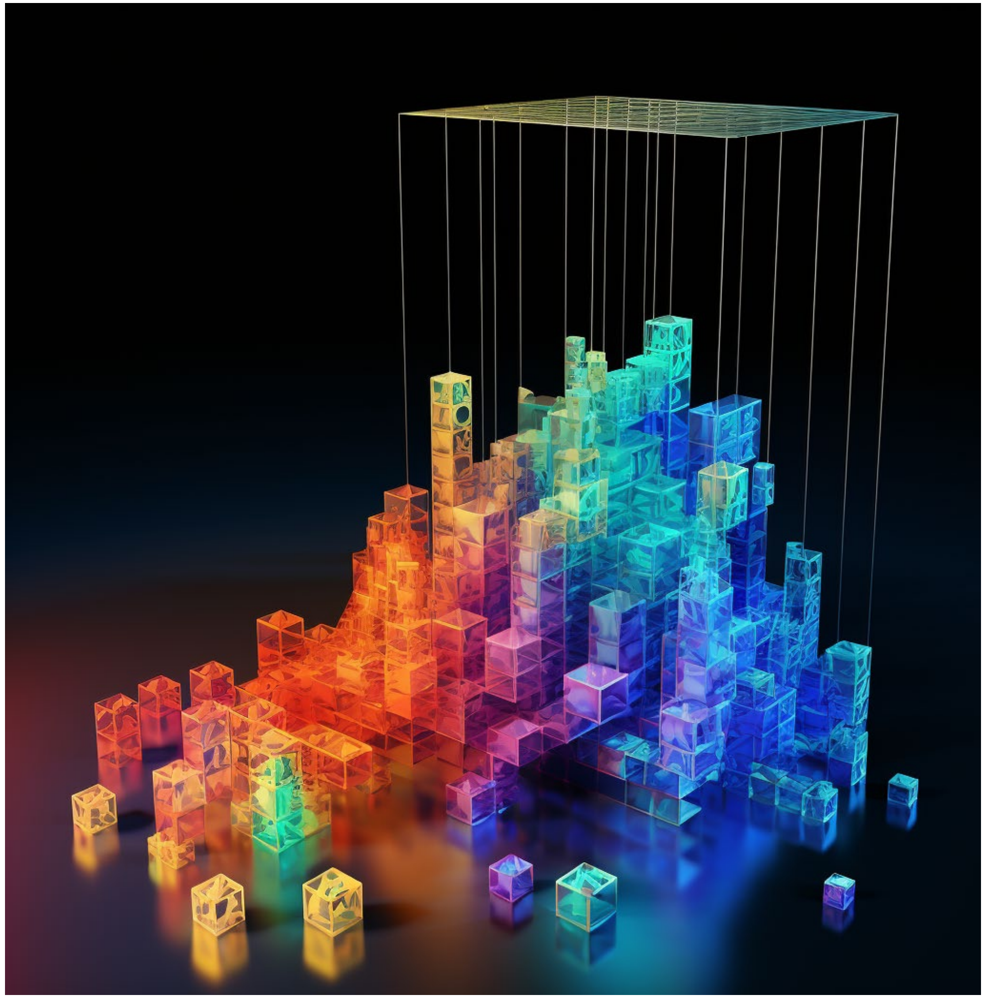
Insurance alternative: Spectral Pricing Rule (risk measure)

- Rather than fixed Z, use X to define "what's bad" and create a custom Z_X
- Define $\rho(X) = \max_{Z \text{ in } \mathcal{Z}} E[XZ]$, the worst risk-adjusted outcome over many Z
- $\rho(X) = E[XZ_X]$ for some Z_X in $\mathcal{Z}$, a customized **contact function** state price density measuring how much we care about each size of loss
- Hardy-Littlewood: X and Z_X must be comonotonic (increase together)
- The set $\mathcal{Z}$ of acceptable Z can be defined from a **distortion function** g, an increasing, concave function $[0, 1] \to [0, 1]$, see PIR 10.9, p.261

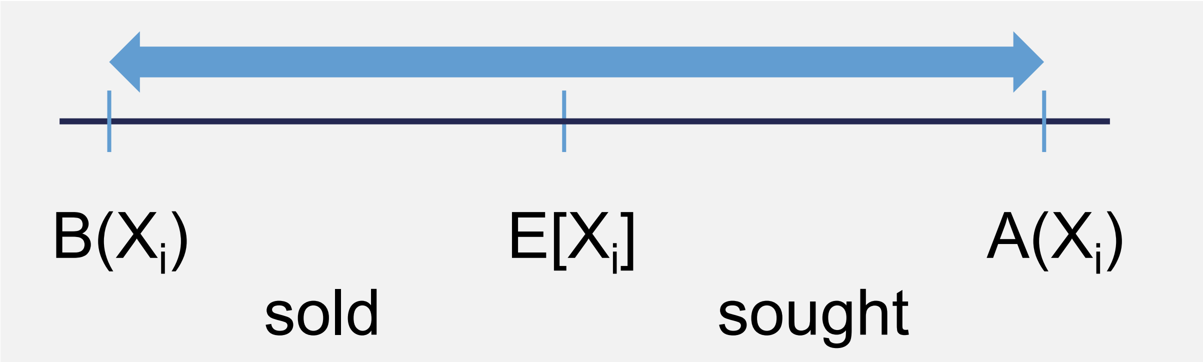
Spectral Pricing Rules have a positive bid-ask spread

- If $\mathcal{Z}$ is large enough, then $\rho(X) = E[XZ_X] > E[X]$ because we can find a Z_X that weights bad (large) outcomes more than small ones
- Interpret $\rho(X) = A(X)$ as the **ask** price
- **Bid** simply $B(X) = -A(-X) = \min_Z E[XZ]$
- $\min_Z E[XZ] = B(X) < E[X] < A(X) = \max_Z E[XZ]$
- Spread $A(X) - B(X)$ is positive



The (Linear) Natural **Premium** Allocation

- $\rho(X) = E[XZ_X]$ total premium
- If $X = X_1 + \dots + X_n$ then it is **natural** to allocate premium $E[X_i Z_X]$ to unit i
- Need to be careful Z_X is unique
- In general $E[X_i Z_X] = E[X_i\, g'(S(X))]$, see PIR Theorem 3, p.261
- The same approach as co-TVaR



- **Natural allocation lies between stand-alone bid and ask prices**
- X_i comonotonic with X, NA = ask
  → X_i is a pure insurance risk
- X_i anti-comonotonic with X, NA = bid
  → X_i is a pure financing risk

What shall we quote?

Spectral ask price for insurance cash flows X_1, X_2

Scenario	X1	X2	X	P	S(X)
3	22	0	22	0.1	0.9
2	28	0	28	0.1	0.8
0	36	0	36	0.1	0.7
1,4,5,6	34	6	40	0.4	0.3
7	45	10	55	0.1	0.2
8	25	40	65	0.1	0.1
9	25	75	100	0.1	0

- Collapse outcomes by value of X, E[• | X] and sort
- $S(x) = \Pr(X > x)$

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Spectral ask price for insurance cash flows X_1, X_2

Scenario	X1	X2	X	P	S(X)	g(S)	Q=diff g(S)
3	22	0	22	0.1	0.9	0.974599	0.025401
2	28	0	28	0.1	0.8	0.923257	0.051342
0	36	0	36	0.1	0.7	0.853469	0.069788
1,4,5,6	34	6	40	0.4	0.3	0.433881	0.419588
7	45	10	55	0.1	0.2	0.299491	0.13439
8	25	40	65	0.1	0.1	0.154702	0.144789
9	25	75	100	0.1	0	0	0.154702

- Collapse outcomes by value of X, E[• | X] and sort
- $S(x) = \Pr(X > x)$
- **Select** dual distortion $g(s) = 1 - (1 - s)^{1.59515}$
- Calibrated to 15% return with assets a = 100
- No default
- Z = Q / P

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Spectral ask price for insurance cash flows X_1, X_2

Scenario	X1	X2	X	P	S(X)	g(S)	Q=diff g(S)
3	22	0	22	0.1	0.9	0.974599	0.025401
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0	36	0	36	0.1	0.7	0.853469	0.069788
1,4,5,6	34	6	40	0.4	0.3	0.433881	0.419588
7	45	10	55	0.1	0.2	0.299491	0.13439
8	25	40	65	0.1	0.1	0.154702	0.144789
9	25	75	100	0.1	0	0	0.154702

EP	31.7	14.9	46.6
EQ	32.31	21.256	53.565
LR	0.9811	0.701	0.87

EP = loss cost
EQ = risk-loaded premium
Sum-products with P and Q columns

- Overall loss ratio is 87.0% (market assumption)
- Non-cat ask price 98.1% loss ratio (no expenses)
- Cat ask price 70.1% loss ratio

- Collapse outcomes by value of X, E[• | X] and sort
- $S(x) = \Pr(X > x)$
- **Select** dual distortion $g(s) = 1 - (1 - s)^{1.59515}$
- Calibrated to 15% return with assets a = 100
- No default
- $Z = Q / P$
- Details: PIR Algos 11.1.1 p.271 and 15.1.1, p.397

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Spectral calculations with financing cash flows X_3, X_4

Scenario	X3	X4	Financing
3	43	35	78
2	37	35	72
0	29	35	64
1,4,5,6	25	35	60
7	10	35	45
8	0	35	35
9	0	0	0

Expected	21.9	31.5	53.4
Price	16.84935	29.58543	46.43478
Return	0.299753	0.064713	0.15

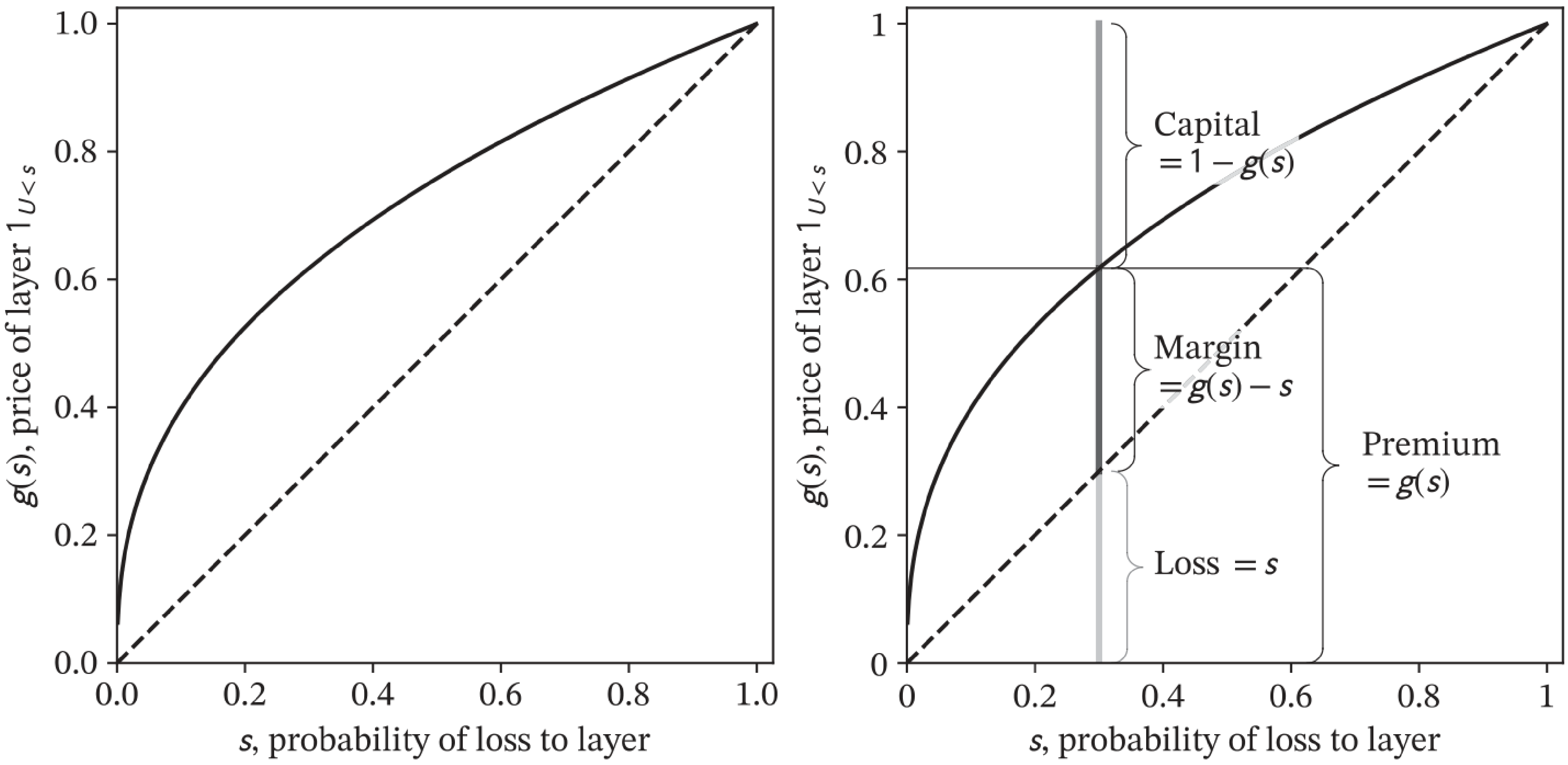
- Overall target return 15% (market)
- X_3 equity has 30% target return
- X_4 agg stop cat bond, a 6.5% return

- Bid price: sort in descending order
- Expected value of t = 1 flow (EP)
- Price is minimum acceptable bid at t = 0 for cash flows made at t = 1 (EQ)
- Price column also equals $\min_Z E[X_i Z]$
- Return = Expected value / Price – 1
- Achieves 15% overall target return
- **Implied ceded loss ratio: 64.6%**

Financing distinct from asset risk!

Parameterization

Distortion function: g(s) = ask price for Bernoulli 0/1 risk



Graphic: Pricing Insurance Risk, Mildenhall & Major (2022), Wiley

Calibrate g to 15% return: five usual suspect distortions

	LR		
unit	X1	X2	total
distortion			
ccoc	102.8%	65.5%	87.0%
ph	101.7%	66.5%	87.0%
wang	100.1%	68.0%	87.0%
dual	98.1%	70.1%	87.0%
tvar	95.7%	72.9%	87.0%

	ROI		
unit	X3	X4	total
distortion			
ccoc	15.0%	15.0%	15.0%
ph	21.0%	11.2%	15.0%
wang	25.0%	8.9%	15.0%
dual	30.0%	6.5%	15.0%
tvar	34.9%	4.3%	15.0%

- PIR §11.3 for a description of the constant cost of capital (CCoC), proportional hazard, Wang, dual, and TVaR distortions
- CCoC is most sensitive to tail risk; TVaR is most sensitive to body risk (volatility)
- Sensitivities consistent with implied loss ratios (insurance) or returns (financing)

Calibrate g to 15% return: five usual suspect distortions

LR			
unit	X1	X2	total
distortion			
ccoc	102.8%	65.5%	87.0%
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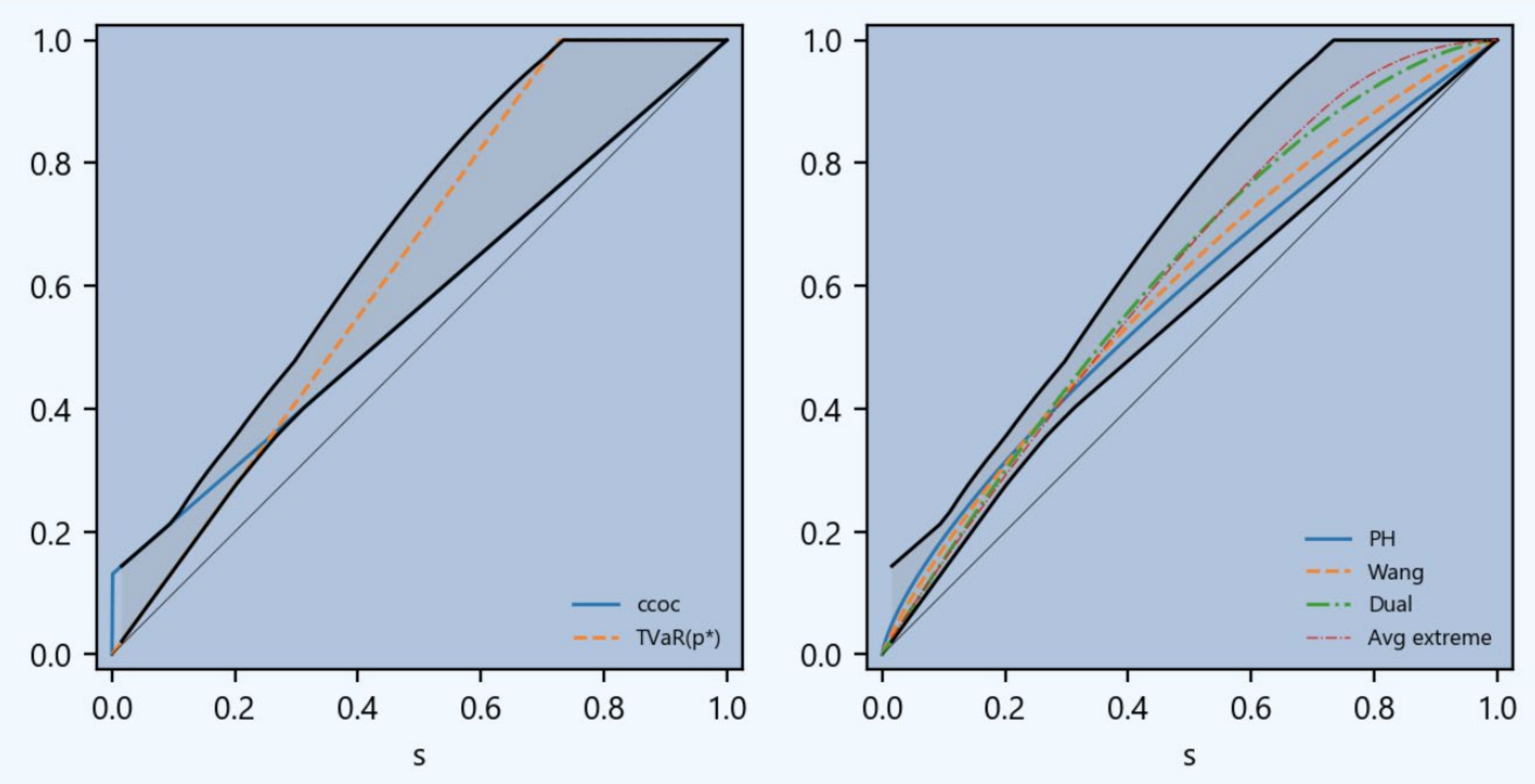
ROI			
unit	X3	X4	total
distortion			
ccoc	15.0%	15.0%	15.0%
ph	21.0%	11.2%	15.0%
wang	25.0%	8.9%	15.0%
dual	30.0%	6.5%	15.0%
tvar	34.9%	4.3%	15.0%

- CCoC: negative margin for non-cat unit X_1, cat unit X_2 very expensive
- TVaR: more balanced, positive margins for both lines
- X_4 cat cover value declines with distortion body-centricity
- X_3 cost of equity increases with distortion body-centricity

Calibrate g to 15% return: five usual suspect distortions



- Shaded area shows all possible distortions
- Left plot: CCoC and TVaR, extreme tail and body sensitivity
- Right plot: PH, Wang, Dual

Applications and Implementation

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Application 1: Diversifying Cat Risk

- Margin for diversifying cat balances two effects
  - Insurance risk in body: ask price, +ve margin
  - Financing benefit in tail: bid price, -ve margin
  - Net price depends on relative weighting of body/volatility and tail capital, captured by g
  - Possible to decompose explicitly

- Model price sensitive to risk appetite

- Default (thoughtless) constant cost of capital is very tail-centric and rarely reflects risk appetite

A diversifying cat is a catastrophe risk from a non-peak peril, such as Chile, Australia or New Zealand.

Application 2: Reinsurance Decision Making

- Many different distortions are consistent with given gross pricing, each corresponding to a different risk appetite
- Can determine ranges for allocated pricing by unit or ceded/net
- **The range of outcomes often brackets typical market cat pricing, showing risk appetite is material to reinsurance decision making!**
- See presentation Mildenhall Lloyd's Oasis Presentation or contact me for more information

Implementation: Aggregate

- All methods described here are easy to implement using `aggregate`, an open-source Python library
- Colab (Jupyter Lab) Notebook to reproduce all exhibits is available
  - Colab Notebook
  - Free to run online with no local installation!
  - Google account required

https://aggregate.readthedocs.io/en/latest/

Link to this presentation

Contact Information & Resources

Contact Information and Resources

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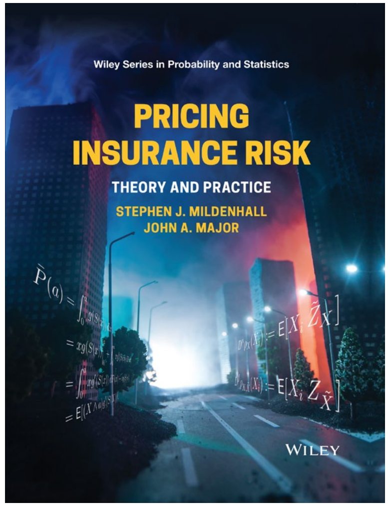
Biography

Stephen Mildenhall is an FCAS with a distinguished 30-year career in insurance and academia. He leads Analytics at QualRisk, focusing on risk and capital optimization in insurance and financial services. Previously, he was an Assistant Professor at St. John's University, New York, and held leadership positions at Aon, including Global CEO of Analytics and head of Aon Reinsurance Analytics. His career began in pricing at Kemper Insurance and CNA, focusing on personal, commercial, and reinsurance lines. At QualRisk, he continues to engage in bespoke consulting projects, while also programming the `aggregate` Python package and contributing to the literature in his field.



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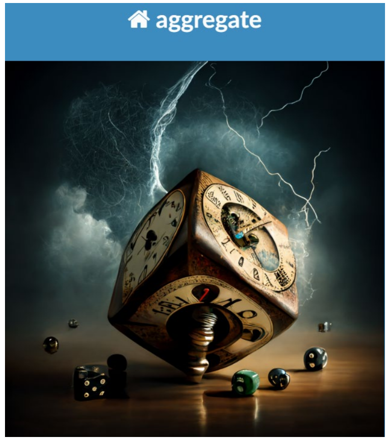
- Case study exhibits
- Supplemental exhibits
- Presentations
- Errata



Software documentation

https://aggregate.readthedocs.io/en/latest/

Code

https://www.github.com/mynl.aggregate

Appendix: Additional Details

Spectral pricing rules have many other nice properties

Pricing rule properties

a) **Monotone**: X ≤ Y implies that ρ(X) ≤ ρ(Y)

b) **Sub-additive**: respects diversification: ρ(X + Y) ≤ ρ(X) + ρ(Y)

c) **Comonotonic additive**: no credit when no diversification. If outcomes X and Y imply same event order, then ρ(X + Y) = ρ(X) + ρ(Y)

d) **Law invariant**: ρ(X) depends only on the distribution of X; no categorical line CoC

A **spectral risk measure** (SRM) ρ(X) is characterized by (a)-(d). They have four representations:

1. Weighted average of VaRs
2. Weighted average of TVaRs
3. Worst over a set of probability scenarios, max { E[XZ] | Z in $\mathcal{Z}_g$}
4. Distorted expected value

$$\rho_g(X) := \int_0^\infty g(S_X(x))\,dx \quad = \mathsf{E}[Xg'(S(X))]$$

See: PIR Theorem 3, p.261

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The Switcheroo: Can exchange X_i and $E[X_i \mid X]$

- $E[X_i \mid X]$ is a random variable: $E[X_i \mid X](\omega) = E[X_i \mid X=X(\omega)]$
- Reduces multi-dimensional problem to one dimension
- $E[X_i Z] = E[E[X_i Z \mid X]] = E[E[X_i \mid X] Z]$
  - Having arranged all Z to be functions of X (**linear** natural allocation)
- Stand-alone price of X_i and $E[X_i \mid X]$ are equal
- Linear natural allocation to X_i and $E[X_i \mid X]$ are equal
- For simulations with distinct X values, $E[X_i \mid X] = X_i$

$E[X_i \mid X]$ and the natural allocation

- Have seen the natural allocation to X_i lies between stand-alone bid and ask prices for X_i, in fact more is true:
  - If $E[X_i \mid X]$ is comonotonic with X, then natural allocation equals $A(E[X_i \mid X])$
    - Pure risk transfer
  - If $E[X_i \mid X]$ is anti-comonotonic with X, then natural allocation equals $B(E[X_i \mid X])$
    - Pure financing

- Easier to meet, test, and see conditions on $E[X_i \mid X]$ than X_i

- If X_i are all thin-tail then $E[X_i \mid X]$ increases with X (Effron's theorem)
  - Ideal insurance situation, most effective diversification

Decomposing the natural allocation price

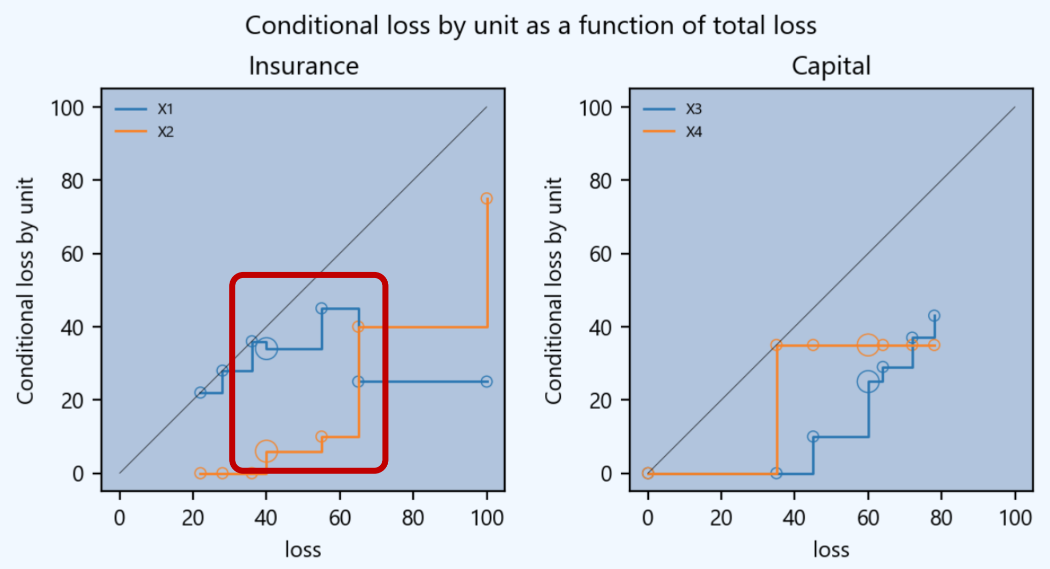
- Can decompose $E[X_i \mid X]$ into $X_i^+ - X_i^-$ where X_i^+, X_i^- are comonotonic with X
- Produces a split $NA(X_i) = E[(X_i^+ - X_i^-)Z] = A(X_i^+) - A(X_i^-)$

  $A(X_i^+)$ = insurance cost with a positive margin

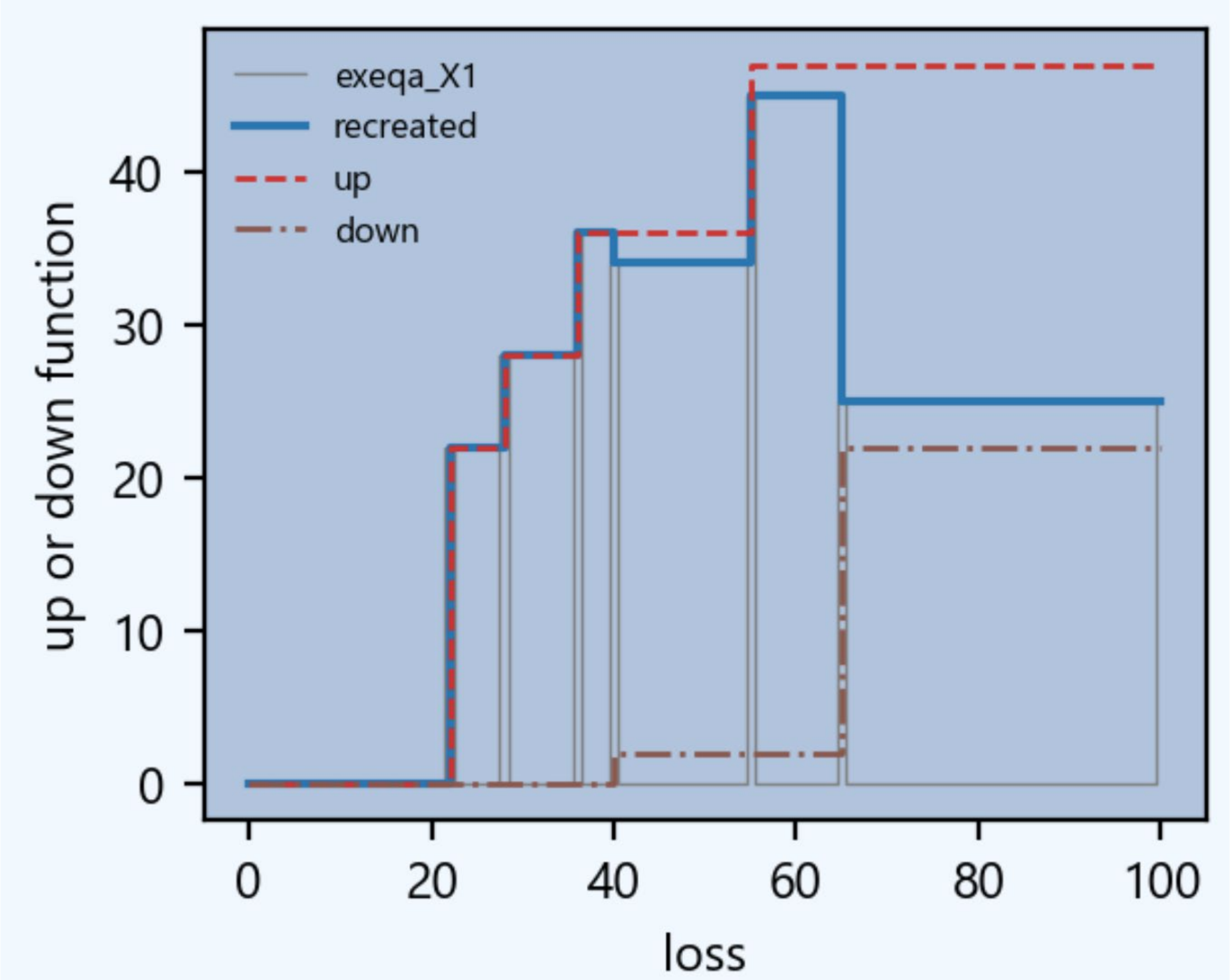
  $-A(X_i^-)$ = financing benefit from selling the capital benefit of X_i, negative margin
- Applies to X_1 but not X_2 which is comonotonic with X

		lna	sa	proj_sa	up	down	umd
unit							
X1	el	31.700	31.700	31.700	37.100	5.400	31.700
	bid	31.621	28.999	29.273	34.288	2.667	31.621
	ask	32.310	34.288	34.084	40.006	7.697	32.310

Key lna: linear natural allocation; sa = stand-alone, proj_sa = stand-alone $E[X_i \mid X]$; up=X^+, down=X^-, umd = up price minus down; Insurance (up) margin = 40.0 – 37.1 = 2.9; financing (down) offset = 7.7 – 5.4 = 2.3, net 2.9 – 2.3 = 0.6; net lna margin = 32.3 – 31.7 = 0.6.



Decomposing the natural allocation price (details)



- Decomposition is not always possible in theory, but it is in practice.
- exeqa_X1 = E[X1 | X]